Approximately dual frame pairs in Hilbert spaces and applications to Gabor frames

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November 21, 2008

Abstract

We discuss the concepts of pseudo-dual frames and approximately dual frames, and illuminate their relationship to classical frames. Approximately dual frames are easier to construct than the classical dual frames, and might be tailored to yield almost perfect reconstruction.

For approximately dual frames constructed via perturbation theory, we provide a bound on the deviation from perfect reconstruction. An alternative bound is derived for the rich class of Gabor frames, by using the Walnut representation of the frame operator to estimate the deviation from equality in the duality conditions.

As illustration of the results, we construct explicit approximate duals of Gabor frames generated by the Gaussian; these approximate duals yield almost perfect reconstruction. Amazingly, the method applies also to certain Gabor frames that are far from being tight.

1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Given a frame $\{f_k\}$ for $\mathcal{H}$, it is well known that there exists at least one dual frame $\{g_k\}$, *i.e.*, a frame for which

$$f = \sum \langle f, f_k \rangle g_k, \quad \forall f \in \mathcal{H}.$$

Unfortunately, it is usually complicated to calculate a dual frame explicitly. Hence we seek methods for constructing *approximate* duals. The current

AMS MSC: 42C15. **Key Words:** Frames, approximate duals, Gabor frames, Gaussian.

paper treats the concepts of pseudo-dual and approximately dual frames and examines their properties. The idea of looking at approximately dual frames has appeared several times in the literature, for example [1, 8, 11] in the wavelet setting and [7] for Gabor systems, but our contribution is the first systematic treatment.

An approximately dual frame $\{g_k\}$ associated with $\{f_k\}$ satisfies

$$\|f - \sum \langle f, f_k \rangle g_k\| \leq \epsilon \|f\|, \quad \forall f \in \mathcal{H}, \tag{1}$$

for some $\epsilon < 1$. (This bound is particularly interesting when ϵ is small.) We will show that any approximately dual frame generates a natural dual frame in the classical sense; further we obtain a family of frames that interpolates between the approximately dual frame and the "true" dual frame.

We also investigate the use of perturbation theory to construct approximately dual frames. There are situations where it is hard to find a dual frame for a given frame $\{f_k\}$, whereas a frame $\{h_k\}$ close to $\{f_k\}$ can be found for which a dual frame $\{g_k\}$ is known explicitly. We present conditions under which such a frame $\{g_k\}$ is approximately dual to $\{f_k\}$, both in the case where $\{g_k\}$ is an arbitrary dual of $\{h_k\}$ and for the particular case where it is the canonical dual.

General Hilbert space estimates might of course be improved in concrete cases. For Gabor frames in $L^2(\mathbb{R})$ we present a direct way to obtain an inequality of the type (1), based on the Walnut representation of the frame operator [15]; the obtained value ϵ measures the deviation from equality in the Gabor duality conditions.

As an illustration of the method, we construct approximately dual frames for two Gabor frames generated by the Gaussian. In these concrete cases, we obtain approximations of type (1) with very small values for ϵ.

The paper is organized as follows. Section 2 contains a few elementary definitions and results from standard frame theory. In Section 3 we introduce the new concepts of pseudo-dual and approximately dual frames, and investigate their properties. The (apparently) most useful concept of these, approximately dual frames, is studied in more detail in Section 4; in particular, we show how to obtain approximate duals via perturbation theory. The direct estimates for Gabor frames are found in Section 5, and the applications to Gabor frames generated by the Gaussian are in Section 6. Finally, an appendix collects a few examples and proofs related to classical frame theory.

2 Basic frame theory

For the introductory frame material that follows, see any standard reference on frames, such as [5], [10], [16], [2, Chapter 5].

A sequence $\{f_k\}$ in $\mathcal{H}$, indexed by an arbitrary countable index set, is a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \qquad \forall f \in \mathcal{H}. \tag{2}$$

Any numbers A, B such that (2) holds are called *(lower and upper) frame bounds.*

If at least the upper inequality holds, then $\{f_k\}$ is a *Bessel sequence.* For later use, note that if $\{f_k\}$ is a Bessel sequence then so is $\{Wf_k\}$ for any bounded operator $W : \mathcal{H} \to \mathcal{H}$.

Given a Bessel sequence $\{f_k\}$, the *synthesis operator* $T : \ell^2 \to \mathcal{H}$ given by

$$T\{c_k\} = \sum c_k f_k$$

is linear and bounded, with $\|T\| \leq \sqrt{B}$; the series converges unconditionally for all $\{c_k\} \in \ell^2$. The adjoint of T is the *analysis operator* $T^* : \mathcal{H} \to \ell^2$ given by

$$T^*f = \{\langle f, f_k\rangle\}.$$

The *frame operator* is TT^*. The frame operator is particularly useful if $\{f_k\}$ actually is a frame, but it is well defined and bounded if we just assume $\{f_k\}$ is a Bessel sequence.

Given a Bessel sequence $\{f_k\}$, a Bessel sequence $\{g_k\}$ is called a *dual frame* if

$$f = \sum \langle f, f_k\rangle g_k, \quad \forall f \in \mathcal{H}. \tag{3}$$

Condition (3) means that analysis using $\{f_k\}$ followed by synthesis using $\{g_k\}$ yields the identity operator. We adopt the terminology of signal processing and speak then about *perfect reconstruction.*

One choice of dual frame is the *canonical dual frame* $\{(TT^*)^{-1}f_k\}$. When $\{f_k\}$ is redundant, infinitely many other dual frames exist.

Note that if $\{f_k\}$ has the upper frame bound B and $\{g_k\}$ is a dual frame, then an application of the Cauchy–Schwarz inequality on

$$\|f\|^2 = \sum \langle f, f_k\rangle\langle g_k, f\rangle$$

shows that $\{g_k\}$ satisfies the lower frame inequality with bound $1/B$; thus the sequences $\{f_k\}$ and $\{g_k\}$ are indeed frames. For later use, observe that no estimate on the upper frame bound for $\{g_k\}$ can be deduced from the duality condition or from knowledge about the frame bounds for $\{f_k\}$, as Example A.1 in Appendix A shows.

3 Various concepts of duality

In order to apply the dual frame expansion (3) for a given frame $\{f_k\}$, we need to find a dual frame $\{g_k\}$. Unfortunately, it might be cumbersome — or even impossible — to calculate a dual frame explicitly. In the literature one finds only a few infinite dimensional non-tight frames for which a dual has been constructed. This paucity of constructions leads us to seek frames that are "close to dual". We will propose two such concepts.

Suppose that $\{f_k\}$ and $\{g_k\}$ are Bessel sequences in $\mathcal{H}$; we will denote their synthesis operators by $T : \ell^2 \to \mathcal{H}$ and $U : \ell^2 \to \mathcal{H}$, respectively. The operators $TU^* : \mathcal{H} \to \mathcal{H}$ and $UT^* : \mathcal{H} \to \mathcal{H}$ will be called *mixed frame operators.* Note that

$$TU^*f = \sum \langle f, g_k \rangle f_k, \quad f \in \mathcal{H},$$

and

$$UT^*f = \sum \langle f, f_k \rangle g_k, \quad f \in \mathcal{H}.$$

Recall that $\{f_k\}$ and $\{g_k\}$ are dual frames when $TU^* = I$ or $UT^* = I$.

Definition 3.1 *Bessel sequences $\{f_k\}$ and $\{g_k\}$ are said to be*

- ***approximately dual frames*** *if $\|I - TU^*\| < 1$ or $\|I - UT^*\| < 1$,*
- ***pseudo-dual frames*** *if TU^* or UT^* is a bijection on $\mathcal{H}$.*

In each definition, the two given conditions are equivalent, by taking adjoints. Note also that the article [4] uses the term approximate dual with a different meaning than we do here.

We illustrate the pseudo-dual idea by means of a frame characterization.

Theorem 3.2 (Characterization of frames) *Let $\{f_k\}$ be a Bessel sequence in $\mathcal{H}$. Then the following statements are equivalent.*

(a) $\{f_k\}$ is a frame for $\mathcal{H}$.

(b) The frame operator TT^* is a bijection on $\mathcal{H}$.

(c) The synthesis operator T is surjective onto $\mathcal{H}$.

(d) $\{f_k\}$ has a dual frame $\{g_k\}$.

(e) $\{f_k\}$ has a pseudo-dual frame $\{g_k\}$.

The new part of the theorem is the pseudo-dual statement in part (e). We need, however, some information from the proof of the known parts at several instances in our paper, so we include a full proof of Theorem 3.2 in Appendix A.

Part (b) of the theorem shows that every frame is pseudo-dual to itself. The proof of (a)⇒(b) gives even more: in fact, denoting the frame bounds for $\{f_k\}$ by A and B, the result (16) in Appendix A shows that

$$\|I - \frac{2}{A+B}TT^*\| \leq \frac{\frac{B}{A}-1}{\frac{B}{A}+1} < 1, \tag{4}$$

i.e., every frame $\{f_k\}$ is approximately dual to the multiple $\{\frac{2}{A+B}f_k\}$ of itself.

Next we establish relations among the three duality concepts.

Lemma 3.3 (Duality relations) *Let $\{f_k\}$ and $\{g_k\}$ be Bessel sequences in $\mathcal{H}$.*

(i) If $\{f_k\}$ and $\{g_k\}$ are dual frames, then $\{f_k\}$ and $\{g_k\}$ are approximately dual frames.

(ii) If $\{f_k\}$ and $\{g_k\}$ are approximately dual frames, then $\{f_k\}$ and $\{g_k\}$ are pseudo-dual frames.

(iii) If $\{f_k\}$ and $\{g_k\}$ are pseudo-dual frames, then $\{f_k\}$ and $\{g_k\}$ are frames.

(iv) If $\{f_k\}$ and $\{g_k\}$ are pseudo-dual frames and $W : \mathcal{H} \to \mathcal{H}$ is a bounded linear bijection, then $\{f_k\}$ and $\{Wg_k\}$ are pseudo-dual frames.

Proof. Statement (i) is an immediate consequence of the definitions. So is (ii), since $\|I - UT^*\| < 1$ implies UT^* is a bijection, with inverse given by a Neumann series. Statement (iii) follows from Theorem 3.2. For the proof of (iv), observe that the synthesis operator for $\{Wg_k\}$ is $X = WU$; the assumptions of $\{f_k\}$ and $\{g_k\}$ being pseudo-dual frames and W being a bijection imply that $XT^* = WUT^*$ is a bijection. □

The property of being a pair of pseudo-dual frames is significantly weaker than being a pair of dual frames. Nevertheless, the following proposition shows that pseudo-dual frame pairs generate dual frame pairs in a natural fashion.

Proposition 3.4 (Pseudo-duals generate duals) *If $\{f_k\}$ and $\{g_k\}$ are pseudo-dual frames, then $\{f_k\}$ and $\{(UT^*)^{-1}g_k\}$ are dual frames.*

Proof. Assuming that $\{f_k\}$ and $\{g_k\}$ are pseudo-dual frames, we know that $(UT^*)^{-1}$ exists and is bounded. Hence $\{(UT^*)^{-1}g_k\}$ is a Bessel sequence. If we analyze with $\{f_k\}$ and synthesize with $\{(UT^*)^{-1}g_k\}$ then we obtain the identity, since

$$\begin{aligned}\sum \langle f, f_k\rangle (UT^*)^{-1} g_k &= (UT^*)^{-1} \sum \langle f, f_k\rangle g_k \\ &= (UT^*)^{-1} UT^* f \\ &= f.\end{aligned}$$

Thus $\{f_k\}$ and $\{(UT^*)^{-1}g_k\}$ are dual frames. □

The relation of being a pair of pseudo-dual frames is symmetric: if $\{f_k\}$ and $\{g_k\}$ are pseudo-dual frames, then so are $\{g_k\}$ and $\{f_k\}$. The pseudo-dual relation is also reflexive on the set of frames since every frame is pseudo-dual to itself, as remarked after Theorem 3.2. Transitivity of the pseudo-dual relation fails (in Hilbert spaces of dimension at least 2) by Example 4.4 below.

4 Approximately dual frames

In this section we focus on approximately dual frames $\{f_k\}$ and $\{g_k\}$. As before, we denote the associated synthesis operators by T and U, respectively.

Our main goal is to demonstrate how one can use this concept to obtain what in engineering terms would be called "almost perfect reconstruction".

Recall that if $\{f_k\}$ and $\{g_k\}$ are approximately dual frames then

$$\|f - \sum \langle f, f_k \rangle g_k\| = \|(I - UT^*)f\| \leq \|I - UT^*\| \, \|f\|.$$

If $\|I - UT^*\| << 1$ then we see that $\sum \langle f, f_k \rangle g_k$ "almost reconstructs" f, which motivates our terminology.

We first strengthen Proposition 3.4, for approximately dual frames and the associated "natural" dual frame.

Proposition 4.1 *Assume that $\{f_k\}$ and $\{g_k\}$ are approximately dual frames. Then the following hold:*

(i) The dual frame $\{(UT^*)^{-1} g_k\}$ of $\{f_k\}$ can be written

$$(UT^*)^{-1} g_k = g_k + \sum_{n=1}^{\infty} (I - UT^*)^n g_k. \tag{5}$$

(ii) Given $N \in \mathbb{N}$, consider the corresponding partial sum,

$$\gamma_k^{(N)} = g_k + \sum_{n=1}^{N} (I - UT^*)^n g_k. \tag{6}$$

Then $\{\gamma_k^{(N)}\}$ is an approximate dual of $\{f_k\}$. Denoting its associated synthesis operator by Z, we have

$$\|I - ZT^*\| \leq \|I - UT^*\|^{N+1}. \tag{7}$$

Proof. Assuming that $\{f_k\}$ and $\{g_k\}$ are approximately dual frames, the inverse of UT^* can be written via a Neumann series as

$$(UT^*)^{-1} = \big(I - (I - UT^*)\big)^{-1} = \sum_{n=0}^{\infty} (I - UT^*)^n. \tag{8}$$

The result in (i) now follows by applying this expansion to g_k and recalling Proposition 3.4.

For (ii), note that $\{\gamma_k^{(N)}\}$ is a Bessel sequence since it is obtained from the Bessel sequence $\{g_k\}$ by a bounded transformation. And

$$\begin{aligned} ZT^*f &= \sum \langle f, f_k \rangle \big(I + \sum_{n=1}^{N} (I - UT^*)^n \big) g_k \\ &= \sum_{n=0}^{N} (I - UT^*)^n UT^* f \\ &= \sum_{n=0}^{N} (I - UT^*)^n \big(I - (I - UT^*) \big) f \\ &= f - (I - UT^*)^{N+1} f \end{aligned}$$

by telescoping. Thus

$$\begin{aligned} \|I - ZT^*\| &= \|(I - UT^*)^{N+1}\| \\ &\leq \|I - UT^*\|^{N+1} < 1. \end{aligned}$$

□

Remark 4.2 In view of formula (5) in Proposition 4.1, an approximately dual frame $\{g_k\}$ associated with a frame $\{f_k\}$ can be regarded as a zero-th order approximation to the (exact) dual frame $\{(UT^*)^{-1} g_k\}$. In case $\|I - UT^*\|$ is small, reconstruction using the approximate dual $\{g_k\}$ is close to perfect reconstruction. The result in (ii) yields a family of approximately dual frames that interpolates between the approximate dual $\{g_k\}$ and the dual frame $\{(UT^*)^{-1} g_k\}$; the estimate (7) shows that by choosing N sufficiently large, we can obtain a reconstruction that is arbitrarily close to perfect. The drawback of the result with respect to potential applications is the complicated structure of the operator in (6) defining the sequence $\gamma_k^{(N)}$.

We have now presented the basic facts for approximately dual frames. In the following, we will present concrete settings where they yield interesting new insights.

To motivate the results, we note (again) that it can be a nontrivial task to find the canonical dual frame (or any other dual) associated with a general frame $\{f_k\}$. We seek to connect this fact with perturbation theory by asking the following question: if we can find a frame $\{h_k\}$ that is close to $\{f_k\}$ and

for which it is possible to find a dual frame $\{g_k\}$, does it follow that $\{g_k\}$ is an approximate dual of $\{f_k\}$? We will present some sufficient conditions for an affirmative answer.

First we state a general result, valid for dual frames $\{g_k\}$ with sufficiently small Bessel bound; later we state a more explicit consequence for the case where $\{g_k\}$ is the canonical dual frame, see Theorem 4.5. Technically, we need not assume that $\{f_k\}$ is a frame in the general result: the frame property follows as a conclusion. On the other hand, as explained above, the main use of the result is in a setting where $\{f_k\}$ is known to be a frame in advance.

Theorem 4.3 (Dual of perturbed sequence) *Assume that $\{f_k\}$ is a sequence in $\mathcal{H}$ and that $\{h_k\}$ is a frame for which*

$$\sum |\langle f, f_k - h_k \rangle|^2 \leq R \, \|f\|^2, \quad \forall f \in \mathcal{H},$$

for some $R > 0$. Consider a dual frame $\{g_k\}$ of $\{h_k\}$ with synthesis operator U, and assume $\{g_k\}$ has upper frame bound C.

If $CR < 1$ then $\{f_k\}$ and $\{g_k\}$ are approximately dual frames, with

$$\|I - UT^*\| \leq \|U\| \sqrt{R} \leq \sqrt{CR} < 1.$$

Proof. With our usual notation, we have $UV^* = I$ since $\{g_k\}$ and $\{h_k\}$ are dual frames. Hence

$$\|I - UT^*\| = \|U(V^* - T^*)\| \leq \|U\| \, \|V^* - T^*\| \leq \sqrt{CR} < 1.$$

$\square$

It is crucial in Theorem 4.3 that the dual frame $\{g_k\}$ has upper frame bound less than $1/R$. Otherwise $\{g_k\}$ need not be pseudo-dual to $\{f_k\}$, let alone approximately dual, as the next example shows.

Example 4.4 Consider $\mathcal{H} = \mathbb{C}^2$ with the standard basis $\{e_1, e_2\}$. Let $\epsilon > 0$ and consider the frames

$$\{f_k\} = \{0, e_1, e_2\}, \quad \{h_k\} = \{\epsilon e_1, e_1, e_2\}, \quad \{g_k\} = \{\epsilon^{-1} e_1, 0, e_2\}.$$

Write T, V, U for the associated synthesis operators. Note that

$$TU^* f = \langle f, e_2 \rangle e_2;$$

this operator is neither injective nor surjective, and so $\{f_k\}$ and $\{g_k\}$ are not pseudo-dual frames. Clearly, $\{h_k\}$ is a frame for $\mathbb{C}^2$, regardless of the choice of ϵ. Now, because

$$\sum_{k=1}^{3} |\langle f, f_k - h_k \rangle|^2 = |\langle f, \epsilon e_1 \rangle|^2 \leq \epsilon^2 \|f\|^2, \quad \forall f \in \mathbb{C}^2,$$

the condition in Theorem 4.3 is satisfied with $R = \epsilon^2$. Thus we see that no matter how close $\{h_k\}$ gets to $\{f_k\}$ (meaning, no matter how small ϵ is), the frame $\{g_k\}$ is not pseudo-dual to $\{f_k\}$. Note that for $\epsilon < 1$, the frame $\{g_k\}$ has the upper frame bound $C = \epsilon^{-2} = 1/R$; thus, Theorem 4.3 is not contradicted.

In this example it is immediate to see that $TV^* = I$ and $VU^* = I$; thus $\{f_k\}$ and $\{h_k\}$ are dual frames, and so are $\{h_k\}$ and $\{g_k\}$. In particular, the example shows that the pseudo-dual property is not transitive. The same argument applies in any separable Hilbert space of dimension at least 2. $\square$

For practical applications of Theorem 4.3, it is problematic that the Bessel bound C on $\{g_k\}$ might increase when one reduces R by taking $\{h_k\}$ very close to $\{f_k\}$: in this way, it might not be possible to get $CR < 1$ just by taking $\{h_k\}$ sufficiently close to $\{f_k\}$. In the case when $\{g_k\}$ is the *canonical* dual of $\{h_k\}$ the problem does not arise: as the next result shows, if $\{h_k\}$ is sufficiently close to $\{f_k\}$, then the canonical dual $\{g_k\}$ of $\{h_k\}$ is approximately dual to $\{f_k\}$. For the statement of this result we need to assume that $\{f_k\}$ is a frame, since its lower frame bound controls the perturbation hypothesis.

Theorem 4.5 (Canonical dual of perturbed frame) *Let $\{f_k\}$ be a frame for $\mathcal{H}$ with frame bounds A, B. Let $\{h_k\}$ be a sequence in $\mathcal{H}$ for which*

$$\sum |\langle f, f_k - h_k \rangle|^2 \leq R \, \|f\|^2, \quad \forall f \in \mathcal{H},$$

for some $R < A/4$. Denote the synthesis operator for $\{h_k\}$ by V.

Then $\{h_k\}$ is a frame. Its canonical dual frame $\{g_k\} = \{(VV^*)^{-1} h_k\}$ is an approximate dual of $\{f_k\}$ with

$$\|I - UT^*\| \leq \frac{1}{\sqrt{A/R} - 1} < 1,$$

where U denotes the synthesis operator for $\{g_k\}$.

Proof. The sequence $\{h_k\}$ is a frame with frame bounds $(\sqrt{A}-\sqrt{R})^2$ and $(\sqrt{B}+\sqrt{R})^2$, by [2, Corollary 5.6.3]; the proof is short, and so we give it here. We have $\|T^*-V^*\|^2 \leq R$ by assumption, and so

$$\left|\|T^*f\|_{\ell^2} - \|V^*f\|_{\ell^2}\right| \leq \sqrt{R}\|f\|$$

for all $f \in \mathcal{H}$, from which the desired frame bounds follow easily. This part of the proof uses only $R < A$.

The canonical dual frame of $\{h_k\}$ is $\{g_k\} = \{(VV^*)^{-1}h_k\}$, with frame bounds

$$\frac{1}{(\sqrt{B}+\sqrt{R})^2} \quad \text{and} \quad \frac{1}{(\sqrt{A}-\sqrt{R})^2}$$

by [2, Lemma 5.1.6]. In terms of the synthesis operator U for $\{g_k\}$, the upper bound says

$$\|U\| \leq \frac{1}{\sqrt{A}-\sqrt{R}}.$$

Theorem 4.3 therefore implies

$$\begin{aligned} \|I - UT^*\| &\leq \|U\|\sqrt{R} \\ &\leq \frac{\sqrt{R}}{\sqrt{A}-\sqrt{R}} \\ &= \frac{1}{\sqrt{A/R}-1}. \end{aligned}$$

To complete the proof, just notice this last expression is smaller than 1 if and only if $R < A/4$. $\square$

We end this section with remarks concerning the appearance of approximate duals in the literature. Approximately dual frames have been employed in the L^2 theory of wavelets; for example, Holschneider developed a sufficient condition for a pair of wavelet systems to be approximately dual [11, Section 2.13], and Gilbert *et al.* [8] obtained approximate duals for highly oversampled wavelet systems by perturbing the continuous parameter system. Approximate duals have proved central to the recent solution of the Mexican hat wavelet spanning problem [1]. There the synthesizing wavelet frame is given, being generated by the Mexican hat function, and the task is to construct an approximately dual analyzing frame. Interestingly, this

approximate dual construction holds in all L^q spaces (in the frequency domain), which suggests that parts of the theory in this paper might extend usefully to Banach spaces.

Approximately dual frames in Gabor theory arose in the work of Feichtinger and Kaiblinger [7, Sections 3,4]. They studied the stability of Gabor frames with respect to perturbation of the generators and the time-frequency lattice; among many other results, they showed that a sufficiently small perturbation (measured in a specific norm) of a dual frame associated with a Gabor frame leads to an approximately dual frame. In the next sections, we construct *explicit* examples of approximately dual Gabor frames.

5 Gabor frames and approximate duals

The general frame estimates in Section 4 are based on operator inequalities such as the product rule $\|TU\| \leq \|T\|\|U\|$. We will apply these general estimates in the next section to obtain explicit approximate duals for Gabor frames generated by a Gaussian. First, though, we will sharpen the general frame estimates in the concrete, rich setting of Gabor theory. These sharper estimates also will be applied in the next section.

A *Gabor frame* is a frame for $L^2(\mathbb{R})$ of the form

$$\{e^{2\pi imb}g(x-na)\}_{m,n\in\mathbb{Z}}$$

for suitably chosen parameters $a, b > 0$ and a fixed function $g \in L^2(\mathbb{R})$, called the *window function.* The number a is called the *translation parameter* and b is the *modulation parameter.* Introducing the operators

$$(T_a g)(x) = g(x-a),\ (E_b g)(x) = e^{2\pi ibx}g(x), \quad \text{for } a, b, x \in \mathbb{R},$$

the Gabor system can be written in the short form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. For more information on Gabor analysis and its role in time–frequency analysis we refer to the book by Gröchenig [9].

We state now the duality conditions for a pair of Gabor systems, due to Ron and Shen [14]. We will apply the version presented by Janssen [12]:

Lemma 5.1 *Two Bessel sequences $\{E_{mb}T_n\varphi\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ form*

dual frames for $L^2(\mathbb{R})$ if and only if the equations

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \overline{\varphi(x-ak)}g(x-ak) - b &= 0, & (9)\\ \sum_{k\in\mathbb{Z}} \overline{\varphi(x-n/b-ak)}g(x-ak) &= 0, \quad n \in \mathbb{Z}\setminus\{0\}, & (10)\end{aligned}$$

hold a.e.

Recall that the Wiener space W consists of all bounded measurable functions $g : \mathbb{R} \to \mathbb{C}$ for which

$$\sum_{n\in\mathbb{Z}} \|g\chi_{[n,n+1[}\|_\infty < \infty.$$

It is well known that if $g \in W$ then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for each choice of $a, b > 0$.

Theorem 5.2 *Given two functions $\varphi, g \in W$ and two parameters $a, b > 0$, let T denote the synthesis operator associated with the Gabor system $\{E_{mb}T_{na}\varphi\}_{m,n\in\mathbb{Z}}$, and U the synthesis operator associated with $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. Then*

$$\|I - UT^*\| \le \frac{1}{b}\left[\left\|b - \sum_{k\in\mathbb{Z}} \overline{T_{ak}g}T_{ak}\varphi\right\|_\infty + \sum_{n\neq 0}\left\|\sum_{k\in\mathbb{Z}} \overline{T_{n/b}T_{ak}g}T_{ak}\varphi\right\|_\infty\right].$$

Proof. The starting point is the Walnut representation of the mixed frame operator associated with $\{E_{mb}T_{na}\varphi\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. According to Theorem 6.3.2 in [9],

$$UT^*f(\cdot) = \frac{1}{b}\sum_{n\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}} \overline{T_{ak}\varphi(\cdot - n/b)}T_{ak}g(\cdot)\right)T_{n/b}f(\cdot), \quad f \in L^2(\mathbb{R}).$$

Thus,

$$
\begin{aligned}
& \|f - UT^*f\| \\
\leq \;& \Big\| \Big(1 - \frac{1}{b}\sum_{k\in\mathbb{Z}} \overline{T_{ak}\varphi(\cdot)} T_{ak}g(\cdot)\Big) f \Big\| \\
& + \frac{1}{b}\Big\| \sum_{n\neq 0} \Big(\sum_{k\in\mathbb{Z}} \overline{T_{ak}\varphi(\cdot - n/b)} T_{ak}g(\cdot)\Big) T_{n/b}f \Big\| \\
\leq \;& \frac{1}{b}\Big\| b - \sum_{k\in\mathbb{Z}} \overline{T_{ak}\varphi} T_{ak}g \Big\|_\infty \|f\| + \frac{1}{b}\sum_{n\neq 0} \Big\| \sum_{k\in\mathbb{Z}} \overline{T_{n/b}T_{ak}\varphi} T_{ak}g \Big\|_\infty \|f\|,
\end{aligned}
$$

which concludes the proof. □

Observe that the terms appearing in the estimate in Theorem 5.2 measure the deviation from equality in the duality relations in Lemma 5.1, with respect to the $\|\cdot\|_\infty$-norm. In particular, Theorem 5.2 proves the "sufficient" direction of Lemma 5.1, under the stated hypotheses.

6 Applications to Gabor frames generated by the Gaussian

The Gaussian $g(x) = e^{-x^2}$ is well known to generate a Gabor frame whenever $ab < 1$. For the case where the parameters a and b are equal and small, Daubechies has demonstrated that the Gabor frame generated by the Gaussian is almost tight, see [5], p.84–86 and [6], p. 980–982; in particular, the formula (4) yields almost perfect reconstruction.

Regardless of the choice of a and b, no convenient explicitly given expression is known for any of the dual frames associated with the Gaussian, though. We will construct explicit, approximately dual frames associated with the Gaussian, for certain choices of a and b. These approximately dual frames provide *almost* perfect reconstruction; Example 6.2 is particularly interesting because it deals with a frame that is far from being tight, *i.e.*, no easy way of obtaining an approximately dual frame is available.

Example 6.1 Consider the (scaled) Gaussian

$$\varphi(x) = \frac{151}{315} e^{-(x/1.18)^2}. \tag{11}$$

The scaling is introduced for convenience; similar constructions can be performed for other Gaussians as well. It is well known that for this Gaussian, the Gabor system $\{E_{mb}T_n\varphi\}_{m,n\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$ for any sufficiently small value of the modulation parameter $b > 0$; we fix $b = 0.06$ in this example. Denote the synthesis operator for that frame by T.

We will use the results derived in this paper to find an approximately dual Gabor frame. Let $h = B_8$ denote the eighth-order B-spline, centered at the origin. This B-spline approximates the Gaussian φ very well: see Figures 1 and 2. The Gabor system $\{E_{bm}T_nh\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, and by Corollary 3.2 in [3] it has the (non-canonical) dual frame $\{E_{bm}T_ng\}_{m,n\in\mathbb{Z}}$, where

$$g(x) = b\sum_{n=-7}^{7} B_8(x+n) \tag{12}$$

is a linear combination of translated B-splines. This g is supported on $[-11, 11]$ and is constantly equal to b on $\operatorname{supp} B_8 = [-4, 4]$; see Figure 3.

We first apply Theorem 4.3. A numerical calculation based on Theorem 9.1.5 in [2] shows that the functions $\{E_{bm}T_n(\varphi - h)\}_{m,n\in\mathbb{Z}}$ form a Bessel sequence with Bessel bound $R = 0.0006$.

Furthermore, a second application of Theorem 9.1.5 in [2] reveals that $\{E_{bm}T_ng\}_{m,n\in\mathbb{Z}}$ has Bessel bound $C = 1$. Thus the hypotheses of Theorem 4.3 are satisfied. Denoting the synthesis operator for $\{E_{bm}T_ng\}_{m,n\in\mathbb{Z}}$ by U, we deduce that

$$\|I - UT^*\| \leq \sqrt{CR} \leq 0.025.$$

On the other hand, an application of Theorem 5.2 immediately yields the better estimate

$$\|I - UT^*\| \leq 0.0027.$$

In terms of the Gabor systems involved, this conclusion says that

$$\|f - \sum_{m,n\in\mathbb{Z}} \langle f, E_{bm}T_n\varphi\rangle E_{bm}T_ng\| < 0.0027\,\|f\|_2, \quad \forall f \in L^2(\mathbb{R}).$$

That is, analysis with the scaled Gaussian and synthesis with the function g yields almost perfect reconstruction.

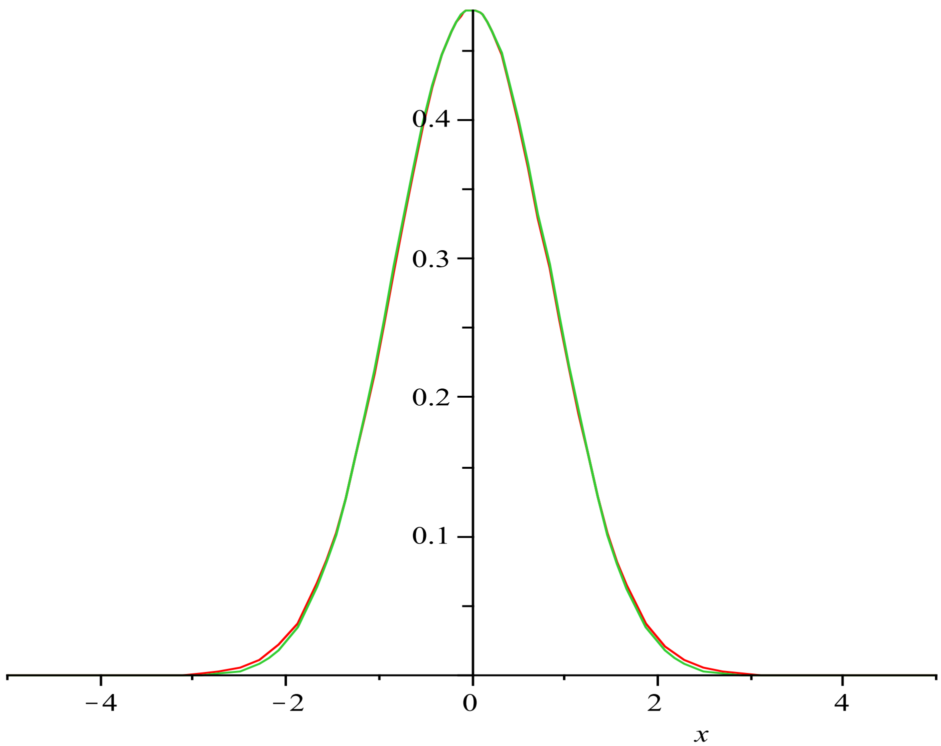

Figure 1: The B-spline B_8 and the scaled Gaussian φ in (11).

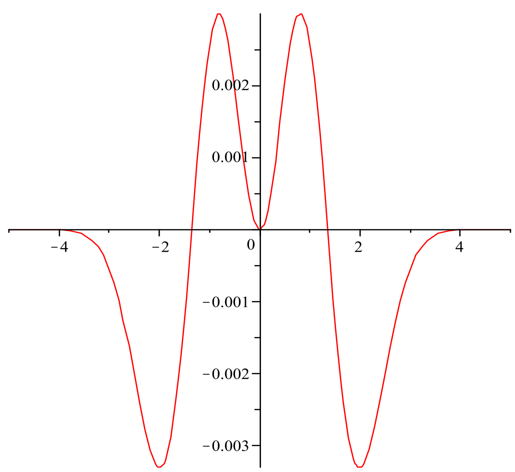

Figure 2: The function $B_8 - \varphi$.

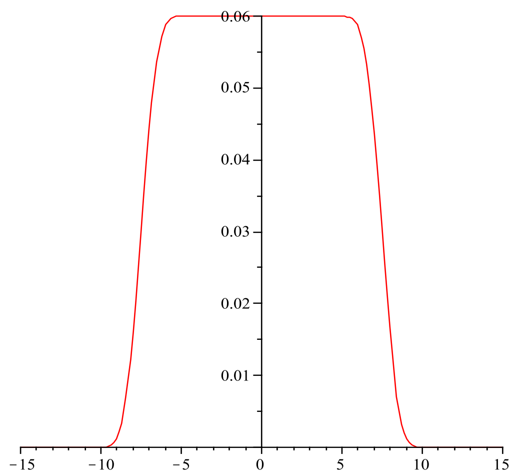

Figure 3: The non-canonical dual window g of B_8, see (12).

We note that the frame considered in this example is almost tight; thus, the approximate dual constructed here does not perform better than using a scaled version of the frame itself as approximate dual. For a frame that is far from being tight this aspect changes drastically in favor of the approximate duals considered in this paper – see Example 6.2. Lastly, we remark that other approximations to the Gaussian are possible too; instead of using B-splines and their duals from [3] we could have used certain other splines and their duals from [13].

$\square$

In the next example we apply Theorem 4.5 and Theorem 5.2. Let us again consider B-spline approximation to a Gaussian. The advantage of using the canonical dual and Theorem 4.5, compared with our use of a non-canonical dual frame in Example 6.1, is that larger values of the modulation parameter b can be handled. Approximating using B_8, our approach in Example 6.1 is restricted to $b \leq 1/15$ (the restriction comes from the underlying results in [3]); as illustration of the larger range for b obtained via Theorem 4.5, we take $b = 0.1$ in the next example.

Example 6.2 Let

$$\varphi(x) = e^{-4x^2}.$$

An application of Theorem 5.1.5 in [2] shows that the functions $\{E_{0.1m}T_n\varphi\}_{m,n\in\mathbb{Z}}$ form a frame for $L^2(\mathbb{R})$ with frame bounds $A = 2.6,\ B = 10.1$. The function

$$h(x) = \frac{315}{151}B_8(2.36x)$$

yields a close approximation of φ, see Figure 4. A numerical calculation based on Theorem 5.1.5 in [2] shows that the functions $\{E_{0.1m}T_n(\varphi-h)\}_{m,n\in\mathbb{Z}}$ form a Bessel sequence with Bessel bound $R = 6.5\cdot 10^{-4} < A/4$.

The function h has support on $[-4/2.36, 4/2.36]$, an interval of length $8/2.36$. Since the modulation parameter $b = 0.1$ is smaller than $(8/2.36)^{-1}$ and the function

$$H(x) = \sum_{k\in\mathbb{Z}} |h(x+k)|^2$$

is bounded above and below away from zero, it follows from Corollary 9.1.7 in [2] that the frame $\{E_{0.1m}T_nh\}_{m,n\in\mathbb{Z}}$ has the canonical dual frame $\{E_{0.1m}T_ng\}_{m,n\in\mathbb{Z}}$, where

$$\begin{aligned} g(x) &= \frac{0.1}{\sum_{n\in\mathbb{Z}}|h(x+n)|^2}\,h(x) \\ &= \frac{15.1}{315}\,\frac{1}{\sum_{n\in\mathbb{Z}}|B_8(2.36(x+n))|^2}\,B_8(2.36x). \end{aligned} \tag{13}$$

See Figure 5.

The frame $\{E_{0.1m}T_ng\}_{m,n\in\mathbb{Z}}$ is approximately dual to $\{E_{0.1m}T_n\varphi\}_{m,n\in\mathbb{Z}}$, by Theorem 4.5. Denoting their synthesis operators by U and T respectively, the approximation rate in Theorem 4.5 is measured by

$$\|I - UT^*\| \le \frac{1}{\sqrt{A/R}-1} \le 0.016. \tag{14}$$

On the other hand, Theorem 5.2 yields the somewhat better estimate

$$\|I - UT^*\| \le \frac{1}{\sqrt{A/R}-1} \le 0.009. \tag{15}$$

Thus the approximate dual frame almost yields perfect reconstruction.

Note that the frame $\{E_{0.1m}T_n\varphi\}_{m,n\in\mathbb{Z}}$ is far from being tight. Thus, the sequence $\{E_{0.1m}T_n\left(\frac{2}{A+B}\,\varphi\right)\}_{m,n\in\mathbb{Z}}$ is a poor approximate dual: the estimate in (4) yields

$$\|I - \frac{2}{A+B}S\| \le \frac{\frac{B}{A}-1}{\frac{B}{A}+1} = 0.59,$$

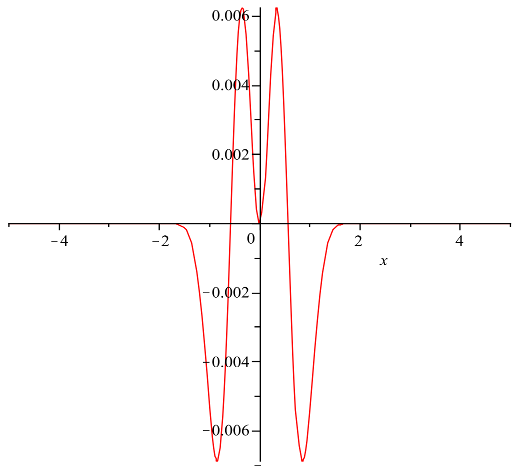

Figure 4: The function $x \mapsto e^{-4x^2} - \frac{315}{151} B_8(2.36x)$.

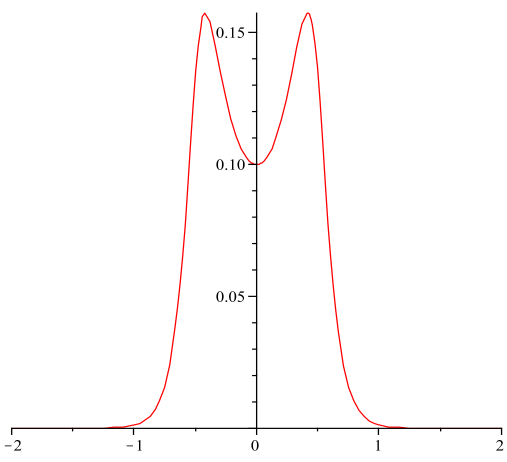

Figure 5: The approximative dual window g in (13).

which is far worse than the result in (15). $\square$

Remark 6.3
1. In case a closer approximation than in (15) is required, an application of the iterative procedure in Proposition 4.1 (ii) can bring us as close to perfect reconstruction as desired. Letting $\{f_k\}$ be a Gabor frame $\{E_{mb}T_{na}f\}_{m,n\in\mathbb{Z}}$ and $\{g_k\}$ be an approximately dual Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, each approximate dual $\{\gamma_k^{(N)}\}$ constructed in Proposition 4.1 will again have Gabor structure. For $N = 1$, an easy calculation shows that $\{\gamma_k^{(N)}\}$ will be the Gabor system $\{E_{mb}T_{na}\gamma\}_{m,n\in\mathbb{Z}}$, where

$$\begin{aligned}\gamma = g + (I - UT^*)g &= 2g - UT^*g \\ &= 2g - \sum_{m',n'\in\mathbb{Z}} \langle g, E_{m'b}T_{n'a}f\rangle E_{m'b}T_{n'a}g \\ &= (2 - \langle g, f\rangle)\, g - \sum_{(m',n')\neq(0,0)} \langle g, E_{m'b}T_{n'a}f\rangle E_{m'b}T_{n'a}g.\end{aligned}$$

For the particular case considered in Example 6.2, and denoting the synthesis operator for $\{E_{mb}T_{na}\gamma\}_{m,n\in\mathbb{Z}}$ by Z, Proposition 4.1 shows that the estimate in (14) will be replaced by

$$\|I - Z^*T\| \leq 0.009^2 = 8.1 \times 10^{-5}.$$

2. Another way of calculating approximate duals for Gabor frames based on $\varphi(x) = e^{-4x^2}$ would be to follow the procedure in Example 6.2 with a function h of the type

$$h(x) = \varphi(x)\chi_I(x)$$

for a sufficiently large interval I, rather than letting h be a B-spline. However, for such a function h the canonical dual window associated with $\{E_{0.1m}T_n\varphi\}_{m,n\in\mathbb{Z}}$ will not be continuous; thus, it will have bad time-frequency properties.

Acknowledgments

Laugesen thanks the Department of Mathematics at the Technical University of Denmark for travel support and a warm welcome.

A Proofs and additional examples

In this Appendix we collect some additional information related to classical frame theory.

First, we present the example announced in Section 2, showing that an estimate on the upper frame bound for a frame $\{g_k\}$ neither can be deduced from the fact that it is dual to a frame $\{g_k\}$, nor from knowledge of the frame bounds for $\{f_k\}$.

Example A.1 Consider the Hilbert space $\mathcal{H} = \mathbb{C}^2$ with the standard orthonormal basis $\{e_1, e_2\}$. Let

$$\{f_1, f_2, f_3\} = \{0, e_1, e_2\}.$$

Then $\{f_k\}$ is a frame with bounds $A = B = 1$. For any $C \in \mathbb{R}$, the sequence

$$\{g_1, g_2, g_3\} = \{Ce_1, e_1, e_2\}$$

is a dual frame of $\{f_k\}$. The (optimal) upper frame bound for $\{g_k\}$ is $C^2 + 1$, which can be arbitrarily large.

Exactly the same considerations apply in an arbitrary separable Hilbert space of dimension at least two. $\square$

We now state the announced proof of Theorem 3.2. The equivalences between the first four statements are well known; we include the proof in order to keep the paper self-contained, and because we refer to some of the steps elsewhere in the paper.

Proof of Theorem 3.2: First recall that T and U are bounded operators from ℓ^2 to $\mathcal{H}$, whenever $\{f_k\}$ and $\{g_k\}$ are Bessel sequences.

(a) implies (b): First, write $S = TT^*$ for the frame operator, and note it is selfadjoint. By part (a) we have $A\|f\|^2 \leq \langle Sf, f\rangle \leq B\|f\|^2$ for all $f \in \mathcal{H}$. That is, $AI \leq S \leq BI$; following the argument by Gröchenig [9] p.91, we infer that $-\frac{B-A}{B+A}I \leq I - \frac{2}{A+B}\, S \leq \frac{B-A}{B+A}I$. Thus,

$$\begin{aligned} \|I - \frac{2}{A+B}S\| = \sup_{\|f\|=1} |\langle (I - \frac{2}{A+B}S)f, f\rangle| \ &\leq \ \frac{B-A}{B+A} \\ &< \ 1. \end{aligned} \tag{16}$$

Therefore $\frac{2}{A+B}S$ is invertible, so that S is a bijection as desired.

(b) implies (c) is immediate.

(c) implies (d): consider the pseudo-inverse $T^\dagger : \mathcal{H} \to \ell^2$. Applying the Riesz representation theorem to the functional $(T^\dagger f)_k = $ (k-th component of $T^\dagger f$), we obtain some element $g_k \in \mathcal{H}$ satisfying $(T^\dagger f)_k = \langle f, g_k \rangle$. The sequence $\{g_k\}$ is Bessel, since

$$\sum |\langle f, g_k \rangle|^2 = \sum |(T^\dagger f)_k|^2 = \|T^\dagger f\|_{\ell^2}^2 \leq \|T^\dagger\|_{\ell^2}^2 \|f\|^2.$$

The analysis operator U^* associated to the sequence $\{g_k\}$ equals exactly the pseudo-inverse $T^\dagger$, by construction, and so $TU^* = TT^\dagger$ equals the identity, by the fundamental property of the pseudo-inverse. Thus $\{g_k\}$ is a dual frame for $\{f_k\}$, and so (d) holds.

(d) implies (e) trivially.

(e) implies (a) as follows: suppose $\{g_k\}$ is a pseudo-dual frame for $\{f_k\}$, so that TU^* is a bijection on $\mathcal{H}$. Then for all $f \in \mathcal{H}$,

$$\begin{aligned} \|f\| &= \|(TU^*)^{-1} TU^* f\| \\ &\leq \|(TU^*)^{-1} T\| \|U^* f\|_{\ell^2}, \end{aligned}$$

and of course $\|U^* f\|_{\ell^2} \leq \|U^*\| \|f\|$. Hence $\{f_k\}$ is a frame with bounds $A = 1/\|(TU^*)^{-1}T\|^2$ and $B = \|U^*\|^2$. $\square$

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